

ON POSITIVE AND ALMOST ALTERNATING LINKS

KAZUHIKO INOUE

ABSTRACT. In this paper, we show that a link which has a positive and almost alternating diagram is alternating, besides that a positive and non-alternating Montesinos link has an almost positive-alternating diagram.

1. INTRODUCTION

A **link** is a disjoint union of circles embedded in \mathbb{S}^3 , and a **knot** is consist of one circle. A **diagram** of a link is a generic projection of a link on \mathbb{S}^2 with over/under information for each double point. A diagram is **alternating** if the over-crossings and under-crossings appear alternately along every component of the diagram, and a link is alternating if it has an alternating diagram. A link diagram is **almost alternating** if one crossing change makes it into an alternating diagram, and a link is almost alternating if it has an almost alternating diagram and no alternating diagram.

A diagram is **positive** if the sign of every crossing is positive. (A negative diagram is the mirror image of a positive diagram.) A link is positive if it has a positive diagram, and a link is **positive and alternating** if it has a positive diagram and an alternating diagram. A link is **positive-alternating** if it has a positive and alternating diagram. Nakamura showed that every positive and alternating link has a positive-alternating diagram ([11]). (We call a positive-alternating diagram, PA-diagram, and a positive and alternating link, PA-link for short.) So our concern is a **positive and almost alternating link**, that is to say a link which has a positive diagram and almost alternating diagram and has no alternating diagram. In section 3, we show the following:

Theorem3.1 Let L an oriented link. If L has a positive and almost alternating diagram then L is alternating.

Besides we know that every positive and almost alternating knot has an almost positive-alternating diagram with up to eleven crossings. Furthermore Jong and Kishimoto showed that every positive knot up to genus two is positive-alternating or almost positive-alternating([7]). A diagram is **almost positive-alternating** if one crossing change makes it into a PA-diagram. We say such diagram almost PA-diagram. In this paper we show the following:

Proposition4.1 Every positive Montesinos link has an almost positive-alternating diagram.

This paper is organized as follows; In **Section 2**, we briefly review almost alternating links and positive alternating links. In **Section 3**, we prove Theorem3.1. In **Section 4**, we characterize positive Montesinos links and prove Proposition4.1.

2. PRELIMINARY

First we shortly introduce some definitions. A diagram D is said to be **equivalent** to a diagram D' if they both represent same link. A diagram is said to be **reduced** if there exists no crossing such that the diagram is separated by splicing the crossing as shown in Figure 1(1). A diagram is said to be **II-reduced** if there are no obvious removal Reidemeister-II move i.e. the link contains no 2-tangle as shown in Figure 1(2) (See [12]).

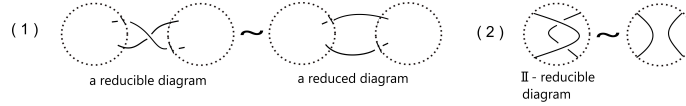


FIGURE 1. A reducible diagram and a II-reducible diagram

A **flype** is an isotopy move applied on a sub tangle of the form $[\pm 1] + t$, and it fixes the endpoints of the sub tangle. See Figure 2. A flype preserves the alternating structure of a diagram. ([8])

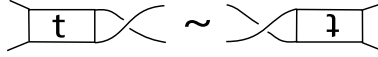


FIGURE 2. A flype

We distinguish a positive tangle from a negative diagram as the following. A tangle is positive if it is as shown in Figure 3(1) and negative as shown in (2). The sign of a crossing point is $+$ if it is as shown in Figure 3(3) and $-$ as shown in (4). A diagram is positive (resp. negative) if every crossing point in the diagram has the same sign $+$ (resp. $-$).

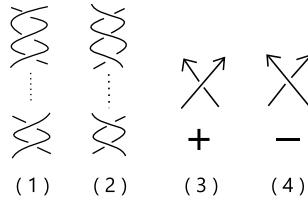


FIGURE 3. A positive(negative) tangle and the sign of a crossing point

Next we introduce some results about almost alternating links. First any alternating link involving a trivial link has an almost alternating diagram. Moreover any alternating link has infinite almost alternating diagrams(see [2]). For example we can make infinite almost alternating diagrams from a trefoil knot as shown in Figure 4. Every diagram turns into an alternating diagram if we change the crossing point d .

Theorem 2.1. *Every positive and almost alternating knot is almost positive-alternating with up to eleven crossings ([4]).*



FIGURE 4. Almost alternating diagrams of a trefoil knot

Theorem 2.2. *Positive knots up to genus two are positive-alternating or almost positive-alternating ([7]).*

Theorem 2.3. *Non-alternating Montesinos links are almost alternating ([1]).*

Theorem 2.4. *Any reduced alternating diagram of a positive alternating link is positive-alternating([11]).*

Then our concern at the moment is the following question.

Question: How is the diagram of a positive and almost alternating link?

In section 3, we show that a positive and almost alternating link does not have a positive and almost alternating diagram for the partial answer to the question above.

3. MAIN THEOREM

Theorem 3.1. *Let L be an oriented link. If L has a positive and almost alternating diagram then L is alternating.*

Proof 3.2. *By the assumption above L has an almost alternating diagram, so we see L is alternating or almost alternating. Our claim is that every positive and almost alternating diagram of L is equivalent to an alternating diagram. First of all if a diagram D is reducible then D is equivalent to an alternating diagram as shown in Figure 5. If D is II-reducible we can see this is equivalent to an alternating diagram in a similar fashion. Therefore we can assume that D is both reduced and II-reduced.*

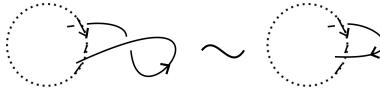


FIGURE 5. A reducible almost alternating diagram

In general positive and almost alternating diagrams are as shown in Figure 6, where the diagram in every shaded portion is positive-alternating.

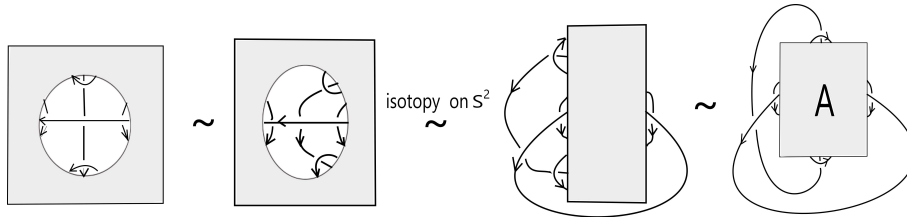


FIGURE 6. positive and almost alternating diagram

The shaded portion in the rightmost figure is equivalent to a disk and we denote this region by A . Note that there does not happen the case as shown in Figure 7, because this diagram is not positive.

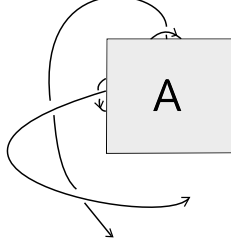


FIGURE 7. This diagram is not positive

Assume disk A separates into disk A_1 and disk A_2 . Since each diagram in A_1 and A_2 is alternating, then the diagram D is equivalent to an alternating diagram D' . (See Figure 8.) Hence L is alternating.

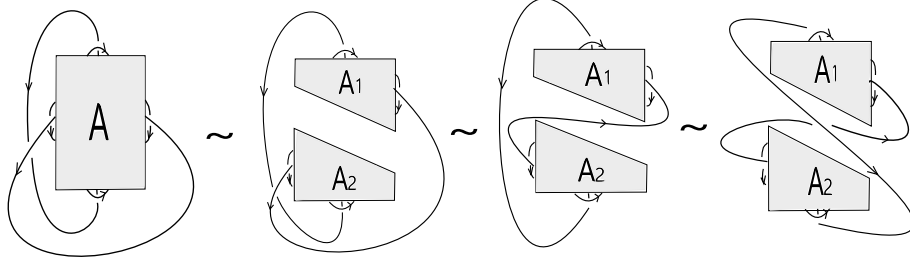


FIGURE 8.

Next we prove that disk A actually separates into disk A_1 and disk A_2 . We name five crossing points outside of A , α , α' , β , β' , d as shown in Figure 9. Besides we also name the strand which passes through α and enters into A , $\overline{\alpha}$, similarly the strand which passes through β and enters into A , $\overline{\beta}$.

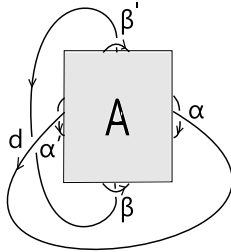


FIGURE 9.

When the strand which passed under the strand $\overline{\alpha}$ at α crosses next strand, there can be three cases as shown in Figure 10 (1) ~ (3).

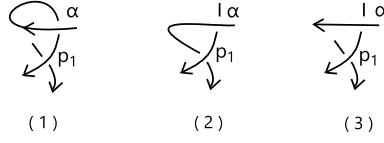
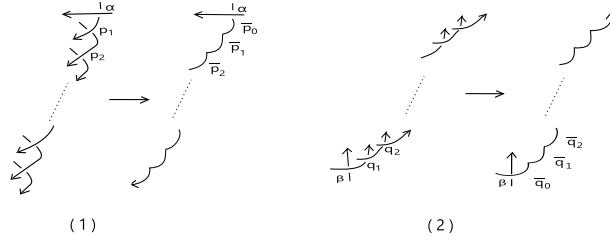


FIGURE 10.

Since the diagram D is positive and alternating in region A therefore in any case the next strand passes under this strand from the right side to the left side as shown in Figure 11(1). We name these crossing points p_1, p_2, \dots and also name the arc from α to p_1 , \bar{p}_0 , from p_1 to p_2 , \bar{p}_1 , similarly $\bar{p}_2, \bar{p}_3, \dots$ and so on. On the other hand we consider the strand which passes over $\bar{\beta}$ as shown in Figure 11(2).

FIGURE 11. The strand which passes under $\bar{\alpha}$ and the strand which passes over $\bar{\beta}$

In the case where a strand crosses a loop or a strand crosses by itself, we regard as shown in Figure 12.

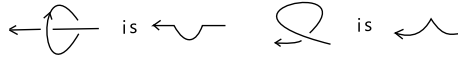


FIGURE 12.

Finally there are two sequences of arcs in A and they are both oriented. This is such as shown in Figure 13(1). If \bar{p}_m and \bar{q}_n cross each other then \bar{p}_m passes over \bar{q}_n from the left side to the right side as shown in Figure 13(2).

We name this crossing point c , then there is a polygon with vertices $\alpha, d, \beta, q_1, q_2, \dots, q_n, c, p_m, p_{m-1}, \dots, p_2, p_1$. And two arcs \bar{p}_m, \bar{q}_n enter this polygon as shown in Figure 14. This is the contradiction to **The Jordan curve theorem**([5]).

Theorem 3.3. (Jordan curve theorem)

Let C be the image of the unit circle, that is $C = \{(x, y); x^2 + y^2 = 1\}$ under an injective continuous mapping γ into \mathbb{R}^2 . Then $\mathbb{R}^2 \setminus C$ is disconnected and consists of two component.

Moreover if \bar{p}_m or \bar{q}_n crosses some arc in $\{\bar{p}_i\}$ or $\{\bar{q}_j\}$ then next it crosses the same arc and enter this polygon again. Because each \bar{p}_i is an arc from under crossing to over crossing and each \bar{q}_j is an arc from over crossing to under crossing. After all we can see that \bar{p}_m and \bar{q}_n never cross each other in A .

For this reason A must separate into A_1 and A_2 hence D is equivalent to an alternating diagram D' . This completes the proof of Theorem 3.1. \square

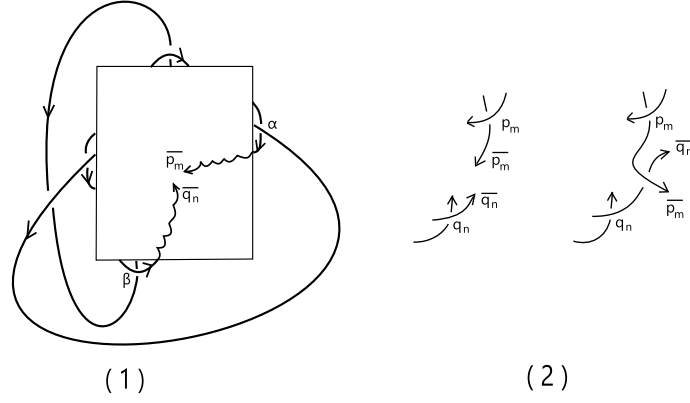
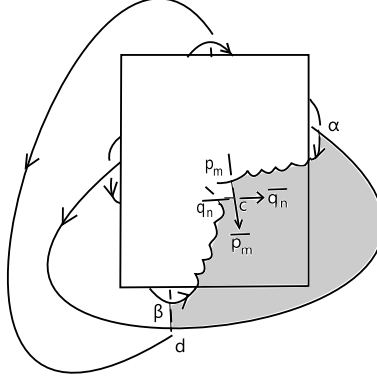
FIGURE 13. The relation between \bar{p}_m and \bar{q}_n 

FIGURE 14.

From the theorem above we know that a positive and almost alternating link has no positive and almost alternating diagram. Furthermore we think the question again. How is the diagram of a positive and almost alternating link? And we give a partial result of this question in section 4.

4. POSITIVE AND ALMOST ALTERNATING MONTESINOS LINK

In this section we would like to study an oriented Montesinos link L denoted by $C(\alpha_1/\beta_1, \alpha_2/\beta_2, \dots, \alpha_n/\beta_n)$, $\alpha_i/\beta_i \in \mathbb{Q}$. Any α_i/β_i represents not only a rational number but also a rational tangle $R_i = (\alpha_i/\beta_i)$. About rational tangles, see [10]. The standard diagram D of L denoted by $D(\alpha_1/\beta_1, \alpha_2/\beta_2, \dots, \alpha_n/\beta_n)$ is shown in Figure 15, where $(\alpha_i/\beta_i) = R_i = R(a_{i1}, a_{i2}, \dots, a_{im})$. That is to say, D is the numerator of the sum of n rational tangles. For example in the case where any $a_{ij} > 0$ R_i is as shown in Figure 15(2) or (3).

Abe and Kishimoto showed that any non-alternating Montesinos link is almost alternating, and we have the following proposition.

Proposition 4.1. *Let L be an oriented Montesinos link and be denoted by*

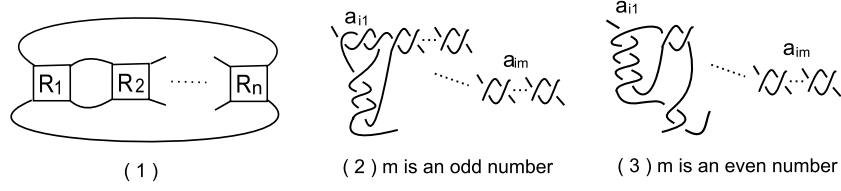


FIGURE 15. The standard diagram of a Montesinos link

$C(\alpha_1/\beta_1, \alpha_2/\beta_2, \dots, \alpha_n/\beta_n)$ where $\alpha_i/\beta_i \in \mathbb{Q}$, and D the standard diagram of L such that $D(\alpha_1/\beta_1, \alpha_2/\beta_2, \dots, \alpha_n/\beta_n)$. If D is positive then L has an almost PA-diagram.

It is to be noted that in general if a link L has a PA-diagram, then L has also an almost PA-diagram. Because we can transform a PA-diagram D of L into an almost PA-diagram D' . See Figure 5.

Before proving the proposition above, we prove two other propositions and one lemma needed later.

Proposition 4.2. *Let L be an oriented link and D be a diagram of L such that $D = D_1 \sharp D_2 \sharp \dots \sharp D_m$ where any D_t is an alternating diagram ($1 \leq t \leq m$). If D is positive, then L has a PA-diagram.*

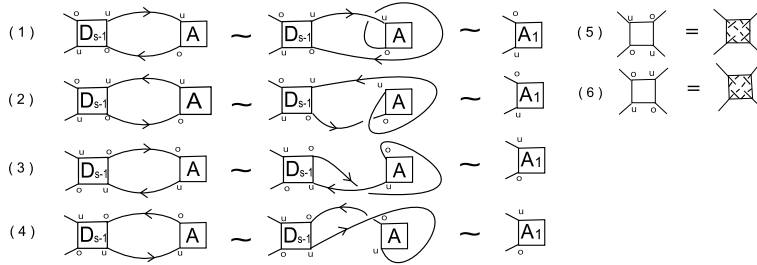
Proof 4.3. *Assume $D = D_1 \sharp D_2 \sharp \dots \sharp D_m$ is positive and $A = D_s \sharp D_{s+1} \sharp \dots \sharp D_m$ ($1 \leq s \leq m$) is alternating as shown in Figure 16. We consider the directions of two arcs on the left-hand side of A and the over/under informations of the leftmost crossings of A and the rightmost crossings of D_{s-1} . Then we can see the four conditions as shown in Figure 17, where the symbol o (resp. u) means that an over-crossing (resp. under-crossing) appears first when we traverse the component from the end point ([1]). By repeating this transformation we can finally obtain a PA-diagram of L as shown in Figure 18.*

□

$$D = \boxed{D_1} \cdots \boxed{D_m} \sim \boxed{D_1} \cdots \boxed{D_{s-1}} \boxed{A}$$

A: PA-diagram

FIGURE 16.

FIGURE 17. Four conditions of $D_{s-1} \sharp A$

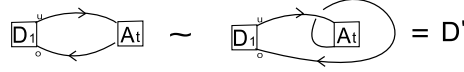
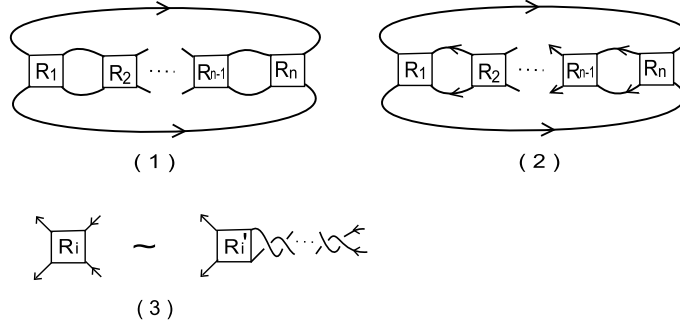


FIGURE 18.

Proposition 4.4. *Let L be an oriented Montesinos link, and D be the standard diagram of L denoted by $D(\alpha_1/\beta_1, \alpha_2/\beta_2, \dots, \alpha_n/\beta_n)$, $\alpha_i/\beta_i \in \mathbb{Q}$. Assume that D is positive, $|\alpha_i/\beta_i| \geq 1$ and $\beta_i \neq 0$ for any i ($1 \leq i \leq n$). Then D is alternating.*

Proof 4.5. *First we consider the case where the directions of left-hand side arcs of R_1 are parallel. In this case, naturally the directions of the right-hand side arcs of R_n are also parallel as shown in Figure 19(1). Besides it is easy to see these directions hold in the case of $R_{n-1}, R_{n-2}, \dots, R_2$ as shown in Figure 19(2). That is to say, the directions of the left-hand side arcs of any tangle R_i are all the same as shown in Figure 19(3). Since each tangle $R_i = R(a_{i1}, a_{i2}, \dots, a_{im})$ is positive and alternating, we know that $a_{ij} \leq 0$ for any j for any j ($1 \leq j \leq m$). Hence $\alpha_i/\beta_i < 0$ for any i ($1 \leq i \leq n$). Then D is necessarily alternating.*

FIGURE 19. The case where the directions of lefthand side arcs of R_1 are parallel

Next we consider the case where the directions of the left-hand side arcs of R_1 are opposite. In this case, the directions of the right-hand side arcs of R_n are as shown in Figure 20(1) or (2). So for any tangle R_i , the directions of the right-hand side arcs are as shown in Figure 20(3) or (4). In any case, we know that $a_{ij} > 0$ for any j ($1 \leq j \leq m$), because any R_i is positive and alternating. Therefore $\alpha_i/\beta_i > 0$ for any i ($1 \leq i \leq n$) and D must be positive. This completes the proof of the proposition. \square

In addition when we meditate upon oriented rational tangles, we can classify them into three types as shown in Figure 21. What is more we can have the next lemma.

Lemma 4.6. *Let R be an oriented rational tangle denoted by (α/β) , where $\alpha/\beta \in \mathbb{Q}_{\neq 0}$, $\beta \neq 0$. If any crossing in R has the same sign $+$, then the following holds.*

- (1) *If R is of type I, then $\alpha/\beta < 0$.*
- (2) *If R is of type II, then $\alpha/\beta > 0$.*
- (3) *If R is of type III and $|\alpha/\beta| \geq 1$, then $\alpha/\beta > 0$.*

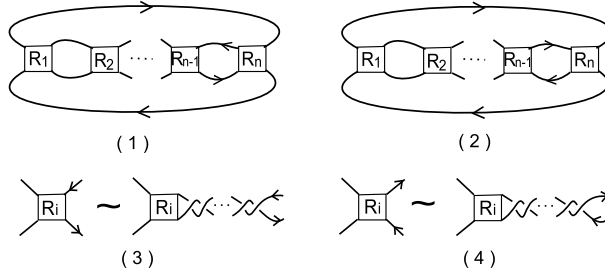


FIGURE 20. The case where the directions of lefthand side arcs of R_1 are opposite

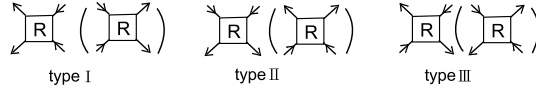


FIGURE 21. Three types of oriented tangles

(4) If R is of type III and $|\alpha/\beta| < 1$, then $\alpha/\beta < 0$.

Proof 4.7. In the case where R is of type I, the oriented tangle R is naturally as shown in Figure 22(1) or (2), and in both cases $\alpha/\beta < 0$. If we reverse all directions, we can prove in exactly the same way. In the case where R is of type II, R is as shown in Figure 22(3) or (4), and it is easy to see in both cases $\alpha/\beta > 0$. Besides, when R is of type III and $|\alpha/\beta| \geq 1$, R is necessarily as shown in Figure 22(5), and $\alpha/\beta > 0$. On the contrary if $|\alpha/\beta| < 1$, R must be as shown in Figure 22(6), and $\alpha/\beta < 0$. We have thus proved the lemma. \square

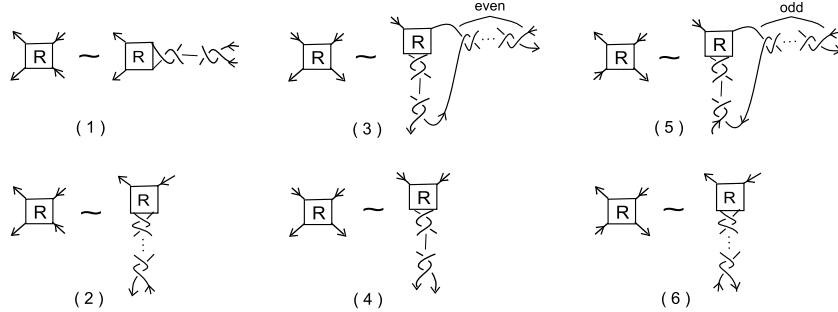


FIGURE 22.

In fact, there are two types in type III as shown in Figure 22(5) and (6). So next we rename type III as shown in Figure 22(5) type III₊, and as shown in Figure 22(6) type III₋. Now we are ready to prove Proposition 4.1.

Proof 4.8. (Proof of Proposition 4.1)

First we consider the case where some $\beta_j = 0$. In this case the tangle α_j/β_j is a ∞ -tangle as shown in Figure 23(1), and the diagram D is like as shown in Figure

23(2), where each $R_k (1 \leq k \leq j-1, j+1 \leq k \leq m)$ is an alternating tangle. So by thinking that the denominator of a tangle R_k is equivalent to a diagram D_k , we can regard $D = D_{j+1} \# \cdots \# D_m \# D_1 \# \cdots \# D_{j-1}$, where any D_k is alternating. Therefore if D is positive then L has a PA-diagram by Proposition 4.2. Hence L has also an almost PA-diagram.

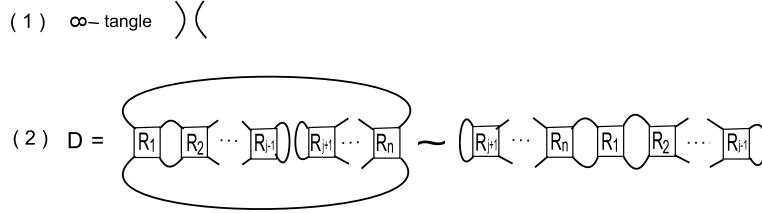


FIGURE 23. The case where some $\beta_j = 0$

Next we consider the case where $\beta_i \neq 0$ for any i ($1 \leq i \leq n$). In this case, from Proposition 4.2 if $|\alpha_i/\beta_i| \geq 1$ for any i ($1 \leq i \leq n$) then D is alternating. Thus if D is non-alternating then there must be some α_j/β_j such that $|\alpha_j/\beta_j| < 1$. That is to say, there exists some tangle R_j such that $R_j = R(a_{j1}, a_{j2}, \dots, a_{jm})$, $a_{jm} = 0$. Furthermore, by Lemma 4.6 we know that there exist some (may be one) rational tangles of type III_- and some (may be one) rational tangles of type II or III_+ in $\{R_i\}$. In this condition we can transform the tangles of type II as shown in Figure 22(3) into such tangles as shown in Figure 24(1) (or all the directions are opposite), and the tangles of type III_+ as shown in Figure 22(5) into the tangles as shown in Figure 24(2) (or all the directions are opposite).

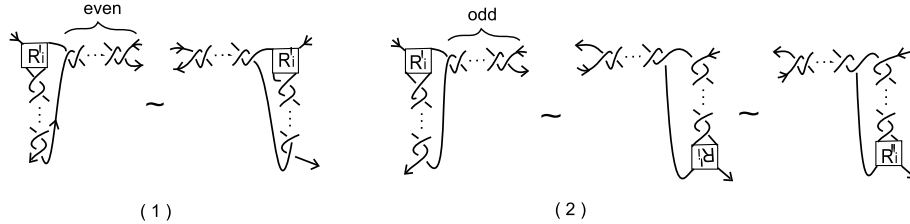


FIGURE 24. The transformation of oriented tangles

After the transformations above, we can regard that D is equivalent to the diagram $D' = D(P_1, P_2, \dots, P_n)$ as shown in Figure 25(1) where $P_k (1 \leq k \leq n)$ is a rational tangle or a 180 degree reversed rational tangle as shown in Figure 25(2) \sim (5). Besides, there exists at least one tangle of type III_- , that is, the tangles as shown in Figure 25(4) or (5). These tangles as shown in Figure 25(2) \sim (5) are all alternating and all crossings in these diagram have the same sign $+$. Namely, these tangles are all PA-tangles. (If a tangle is alternating and every crossing point in this diagram has the same sign $+$, we call this tangle a PA-tangle.) In addition, the depicted symbols of the tangles as shown in (2) and (3) are such as shown in (6) and those of the tangles as shown in (4) and (5) are such as shown in (7).

Assume P_s is the rightmost tangle of type II or type III_+ as shown in Figure 25(2) or (3) in the diagram D' , then all the tangles situated on the right-hand side

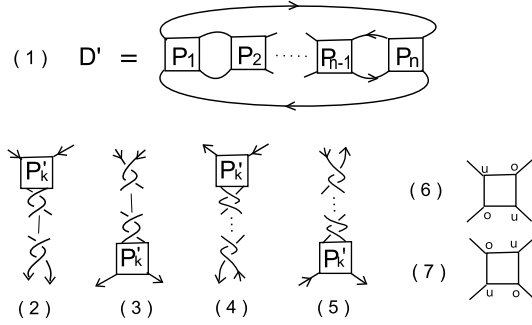


FIGURE 25.

of P_s are of type III_- . Thus when we denote the sum of these tangles by T_0 , T_0 must be a PA-tangle and the depicted symbols of T_0 is as shown in Figure 25(7). Thereby we can transform the diagram D' as shown in Figure 26, and have a diagram D'' which is equivalent to D' where T_1 is a PA-tangle.

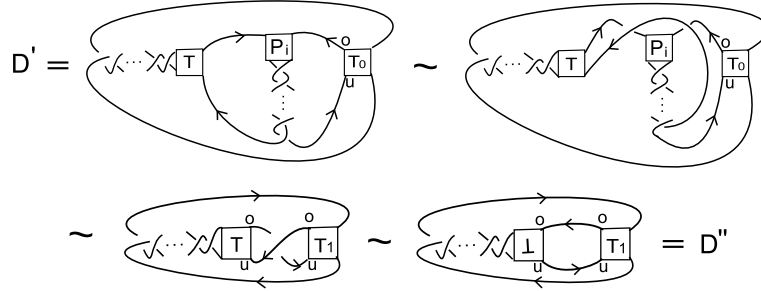


FIGURE 26. The transformation of diagram D

The directions of outer arcs of T_1 are as same as those of T_0 , and it is obvious that if tangle P_{s-1} which is on the lefthand side of T_1 is like as shown in Figure 25(4) or (5) then the tangle sum $P_{s-1} + T_1$ is a PA-tangle. On the other hand, if P_{s-1} is as in Figure 25(2) or (3), we can obtain a PA-tangle in a similar fashion like above. Therefore when v is the number of tangles like as shown in Figure 25(2) or (3), by using v time operations like above we can gain a positive diagram D' as shown in Figure 27 where T_v is a PA-tangle. Moreover we can gain a PA-diagram D''' by changing over/under information of a crossing d . Hence it is clear that L has an almost PA-diagram. This completes the proof of Proposition 4.1. \square

REFERENCES

- [1] T. Abe and K. Kishimoto, The dealternating number and the alternation number of a closed 3-braid, J.Knot Theory Ramifications, **19** (2010), no9, 1157-1181.
- [2] C. Adams, J. Brock, J. Bugbee, T. Comar, A. Huston, A. Joseph and D. Pesikoff, Almost alternating links, Topology Appl. **46** (1992), 151-165.
- [3] C. Bankwitz, Über die torsionzahlen der alternierenden knoten, Math, Ann, **103**, (1930), 145-161.
- [4] P.R. Cromwell, Homogeneous links, J.London Math. Soc(2) **39** (1989), 535-552.

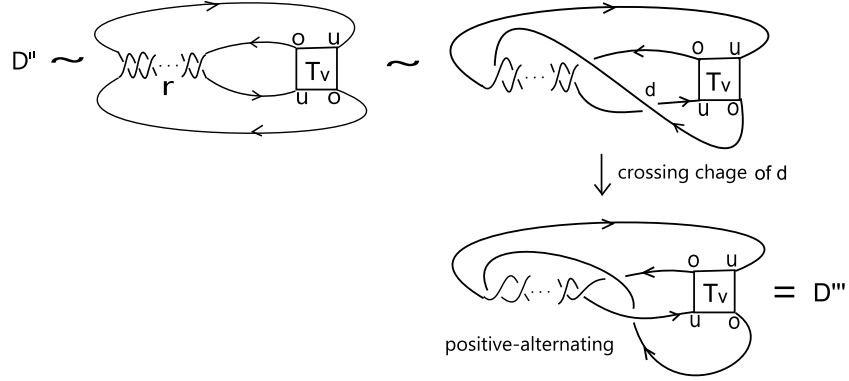


FIGURE 27.

- [5] T. Hales, The Jordan curve theorem, formally and informally, American Mathematical Monthly, **114** (2007), 882-894.
- [6] M. Hirasawa, Triviality and splittability of special almost-alternating diagrams via canonical Seifert surfaces, Topology Appl. **102** (2000), 89-100.
- [7] I. D. Jong and K. Kishimoto, On positive knots of genus two,
- [8] L. H. kauffman and S. Lambropoulon, Classifying and applying rational tangles, Adv. in Appl. Math. **33** (2004), no2, 199-237.
- [9] D. Kim and J. Lee, Some invariants of pretzel links, Bull. Anstral. Math. Soc, vol.75 (2007), 253-271.
- [10] K. Murasugi, Knot theory and its applications, translated by Bohdan Kurpita, (2010).
- [11] T. Nakamura, Positive alternating links are positively alternating, J. Knot Theory ramifications **9** (2000), no1, 107-112.
- [12] T. Tsukamoto, The almost alternating diagrams of the trivial knot, J. Topol. **2**(1) (2009), 77-104.

GRADUATE SCHOOL OF MATHEMATICS, KYUSYU UNIVERSITY, 744, MOTOOKA, NISHI-KU, FUKUOKA, 819-0395, JAPAN